Structural Operational Semantics with First-Order Logic

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Outline

Introduction - Structural Operational Semantics (SOS)

SOS allows for intuitive definition of operational semantics.

Operational semantics typically in terms of transition systems.

Popular for giving semantics to

- programming languages,
- process algebra,
- Petri nets,
- etc.

Introduction - Structural Operational Semantics (SOS)

Semantics is defined with rules.

"If P, then c": $$\frac{P}{c}$$

Set P of premises consists of positive and negative statements.
($x \xrightarrow{a} x'$ and $x \stackrel{a}{\nrightarrow}$)

Conclusion c is a (positive) statement.

Introduction - SOS formats

There exist (syntactic) formats.

- ntyft/ntyxt
- PANTH
- RBB-Cool
- etc.

These formats guarantee certain properties.

(Typically congruence of strong bisimilarity.)

Introduction - Example SOS

$$\frac{}{a.x \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} x'}{y + x \xrightarrow{a} x'} \qquad \frac{x \stackrel{a}{\nrightarrow}}{x\surd}$$

Rules are in ntyft/ntyxt format.

Strong bisimilarity is a congruence.

Introduction - SOS and quantifications

A predicate “is in a deadlock”:

> *“We say that a process is in a deadlock [...] if it cannot do any action. That is, if p is such a process, we have that $p \stackrel{a}{\not\rightarrow} q$ and $p \stackrel{a}{\not\rightarrow} \surd$ for every a, q; [...].”*

J.C.M. Baeten and J.A. Bergstra. Processen en procesexpressies. *Informatie*, 30(3):214–222, 1988.

Introduction - SOS and quantifications

A *weak termination predicate* $\surd\!\!\!\surd$:

$$p\surd\!\!\!\surd \Leftrightarrow \begin{cases} \text{(i)} & p \stackrel{\tau}{\not\rightarrow} \text{ and } p\surd, \text{ or} \\ \text{(ii)} & p \stackrel{\tau}{\rightarrow} \text{ and, for each } q,\ p \stackrel{\tau}{\rightarrow} q \text{ implies } q\surd\!\!\!\surd. \end{cases}$$

Luca Aceto and Matthew Hennessy. Termination, deadlock, and divergence. *Journal of the ACM*, 39(1):147–187, 1992.

Introduction - SOS and quantifications

A *semantically divergence* predicate $\Downarrow$:

$$p \downarrow \text{ and } (\text{for each } q,\ p \xrightarrow{\tau} q \text{ implies } q \Downarrow) \text{ imply } p \Downarrow$$

Luca Aceto and Matthew Hennessy. Termination, deadlock, and divergence. *Journal of the ACM*, 39(1):147–187, 1992.

Introduction - SOS and quantifications

M.R. Mousavi and M. Reniers gave a solution.

- Syntax is quite restricted:

$$\exists_{\overrightarrow{x}} \forall_{\overrightarrow{y}} \exists_{\overrightarrow{z}} \frac{\ldots \wedge \ldots \vee \ldots}{t \xrightarrow{a} u}$$

- Unintuitive semantics (via translation to traditional rules).
- Certain other peculiarities.

M.R. Mousavi and M. Reniers. A Congruence Rule Format with Universal Quantification. *Proceedings of SOS'07*, (to appear).

Outline

Traditional SOS - Syntax

Variables $x, y, \ldots$
Functions $f, g, 0, 1, \delta, +, \ldots$

$$T ::= x \mid f(T, \ldots, T)$$

Predicates $P, Q, \xrightarrow{a}, \surd, \ldots$

$$atom ::= P(T, \ldots, T) \mid \neg P(T) \mid \neg P(T, \_) \mid ??$$

Special negation $t \stackrel{a}{\nrightarrow}$ ("there is no u such that $t \xrightarrow{a} u$")

Rule $\dfrac{S}{a}$ (where S is a set of atoms, a a positive atom)

Traditional SOS - Semantics

Derivability of an atom a given set of assumptions A.

$$\vdash \frac{A}{a}$$

if:

- $a \in A$, or
- there is are rule $\frac{S}{b}$ and substitution σ with
  - $a = b\sigma$ and
  - for all $c \in S \vdash \frac{A}{c\sigma}$

Traditional SOS - Semantics

$$\vdash \frac{\emptyset}{a.0 + 0 \xrightarrow{a} 0}$$

$$\Leftarrow \quad \textit{rule}\ \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'}\ \textit{and}\ \sigma = \{x \mapsto a.0, x' \mapsto 0, y \mapsto 0\}$$

$$\vdash \frac{\emptyset}{a.0 \xrightarrow{a} 0}$$

$$\Leftarrow \quad \textit{rule}\ \frac{}{a.x \xrightarrow{a} x}\ \textit{and}\ \sigma = \{x \mapsto 0\}$$

true

Traditional SOS - Semantics

$$\vdash \frac{\{0 \overset{a}{\nrightarrow}\}}{0\surd}$$

$$\Leftarrow \qquad \mathit{rule}\ \frac{x \overset{a}{\nrightarrow}}{x\surd}\ \mathit{and}\ \sigma = \{x \mapsto 0\}$$

$$\vdash \frac{\{0 \overset{a}{\nrightarrow}\}}{0 \overset{a}{\nrightarrow}}$$

$$\Leftarrow \qquad 0 \overset{a}{\nrightarrow}\ \in \{0 \overset{a}{\nrightarrow}\}$$

true

Traditional SOS - Semantics

Negative statements can cause problems:

$$\frac{\neg P}{P}$$

We consider three-valued models $\langle C, U \rangle$.

C is "Certainly true", U is "Unknown".
($(C \cup U)^{-1}$ is "certainly false.")

Above rule as three-valued model: $\langle \emptyset, \{P\} \rangle$

Traditional SOS - Semantics

We write $A \vDash a$ if $a \in A$ and $A \vDash t \not\rightarrow$ if there is no u with $t \rightarrow u \in A$. We also rewrite $A \vDash S$ if $A \vDash s$ for each $s \in S$.

$\langle C, U \rangle$ is a three-valued stable model if

- $a \in C$ if, and only if, $\vdash \dfrac{N}{a}$ for some N with $C \cup U \vDash N$
- $a \in C \cup U$ if, and only if, $\vdash \dfrac{N}{a}$ for some N with $C \vDash N$

(N is a set of negative atoms here.)

Interested in (information-)least three-valued stable model.

Traditional SOS - Well-Definedness

Generally we are only interested in two-valued models.

That is, models $\langle C, U \rangle$ where $U = \emptyset$.

How to determine that a three-valued model is actually two-valued?

Traditional SOS - Well-Definedness

A stratification is a function f of positive atoms to a set S if

- S is well-ordered
  (i.e. there is no infinite sequence $s > s' > \ldots$)
- for all rules $\frac{S}{a}$ and substitutions σ we have that
  - $b \in S$ implies $f(b\sigma) \leq f(a\sigma)$
  - $t \stackrel{a}{\nrightarrow} \in S$ implies $f(t\sigma \xrightarrow{a} u) < f(a\sigma)$ for all u

If there is a stratification, then the model is two-valued.

Traditional SOS - Well-Definedness

$$\frac{}{a.x \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} x'}{y + x \xrightarrow{a} x'} \qquad \frac{x \stackrel{a}{\not\rightarrow}}{x\surd}$$

$$f(t \xrightarrow{a} u) = 0 \qquad f(t\surd) = 1$$

Traditional SOS - Well-Definedness

$$\frac{\neg P}{P}$$

$$f(P) < f(P)$$

Traditional SOS - Well-Definedness

$$\frac{}{P} \qquad \frac{\neg P}{P}$$

$$f(P) < f(P)$$

Least three-valued stable model is $\langle \{P\}, \emptyset \rangle$!

Traditional - Well-Definedness

Why $\leq$ for positives and $<$ for negatives?

Essence is limiting the number of steps in three-valued model definition:

1. $a \in C$ if $\vdash \dfrac{N}{a}$ and $C \cup U \vDash N$.
2. for $\neg b \in N$, $b \in C \cup U$ if $\vdash \dfrac{N'}{b}$ and $C \vDash N'$
3. for $\neg c \in C$, $c \in C$ if ...
4. ...

The $\leq$ for positives is to get $f(b) < f(a)$ for $b \in N$ of $\vdash \dfrac{N}{a}$.

Traditional SOS - Congruence Format

If all rules are of the form:

$$\frac{\{t_i \xrightarrow{a} y_i \ : \ i \in I\} \cup \{t_j \stackrel{a}{\nrightarrow} \ : \ j \in J\}}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

with all $x_1, \ldots, x_n$ and y_i $(i \in I)$ distinct,

then strong bisimilarity is a congruence for all f.
(i.e. if $p_i \leftrightarroweq q_i$ for $1 \leq i \leq n$, then $f(p_1, \ldots, p_n) \leftrightarroweq f(q_1, \ldots, q_n)$)

Outline

Structural Operational Semantics with First-Order Logic

We want quantifications, implications etc.

That is, we want first-order logic formulae.

We also want it to be an extension of traditional SOS.

Finally, we want to lever traditional notions to this setting.

Infinitary First-Order Kleene Logic - Syntax

We want first-order logic in premises:

$$\varphi ::= a \mid \neg\varphi \mid \bigwedge\{\varphi, \ldots\} \mid \forall_x \varphi$$

Other operators are considered sugar:

$\mathrm{true} = \bigwedge \emptyset$, $x \vee y = \neg \bigwedge\{\neg x, \neg y\}$, etc.

Literals atoms or negation of atoms. (I.e. no more $t \overset{a}{\nrightarrow}$.)

Infinitary First-Order Kleene Logic - Semantics

Set A makes φ true ($A \vDash \varphi$):

$A \vDash a$	iff	$a \in A$
$A \vDash \neg\psi$	iff	$A \nvDash \psi$
$A \vDash \bigwedge \Psi$	iff	for all $\psi \in \Psi$, $A \vDash \psi$
$A \vDash \forall_x \psi$	iff	for all term t, $A \vDash \psi[t/x]$

Set A makes φ false ($A \nvDash \varphi$):

$A \nvDash a$	iff	$\neg a \in A$
$A \nvDash \neg\psi$	iff	$A \vDash \psi$
$A \nvDash \bigwedge \Psi$	iff	there is a $\psi \in \Psi$ with $A \nvDash \psi$
$A \nvDash \forall_x \psi$	iff	there is a t with $A \nvDash \psi[t/x]$

FOL-SOS - Syntax

$$\frac{\varphi}{a}$$

FOL-SOS - Syntax

A predicate "is in a deadlock":

"We say that a process is in a deadlock [...] if it cannot do any action. That is, if p is such a process, we have that $p \overset{a}{\nrightarrow} q$ *and* $p \overset{a}{\nrightarrow} \surd$ *for every* a, q*; [...]."*

$$\frac{\bigwedge_{a \in A}(\forall_y(\neg\, x \overset{a}{\rightarrow} y) \wedge \neg\, x \overset{a}{\rightarrow} \surd)}{\delta(x)}$$

FOL-SOS - Syntax

A *weak termination predicate* $\surd\!\!\!\!\surd$:

$$p\surd\!\!\!\!\surd \Leftrightarrow \begin{cases} \text{(i)} & p \overset{\tau}{\nrightarrow} \text{ and } p\surd\text{, or} \\ \text{(ii)} & p \overset{\tau}{\rightarrow} \text{ and, for each } q,\ p \overset{\tau}{\rightarrow} q \text{ implies } q\surd\!\!\!\!\surd. \end{cases}$$

$$\frac{\forall_y(\neg\, x \overset{\tau}{\rightarrow} y) \wedge x\surd}{x\surd\!\!\!\!\surd} \qquad \frac{\exists_y(x \overset{\tau}{\rightarrow} y) \wedge \forall_y(x \overset{\tau}{\rightarrow} y \Rightarrow y\surd\!\!\!\!\surd)}{x\surd\!\!\!\!\surd}$$

FOL-SOS - Syntax

A *semantically divergence* predicate $\Downarrow$:

$p \downarrow$ and (for each q, $p \xrightarrow{\tau} q$ implies $q \Downarrow$) imply $p \Downarrow$

$$\frac{x \downarrow \wedge \forall_y (x \xrightarrow{\tau} y \Rightarrow y \Downarrow)}{x \Downarrow}$$

FOL-SOS - Syntax

Mousavi & Reniers have to write

$$\forall_y \frac{\bigwedge_{a\in A}(x \stackrel{a}{\nrightarrow} y \wedge x \stackrel{a}{\nrightarrow} \surd)}{\delta(x)}$$

for

$$\frac{\bigwedge_{a\in A}(\forall_y(\neg\, x \stackrel{a}{\rightarrow} y) \wedge \neg\, x \stackrel{a}{\rightarrow} \surd)}{\delta(x)}$$

FOL-SOS - Syntax

Mousavi & Reniers have to write

$$\forall_y \frac{x \stackrel{\tau}{\nrightarrow} y \wedge x\surd}{x\not\surd} \qquad \exists_y \forall_z \frac{x \xrightarrow{\tau} y \wedge (x \stackrel{\tau}{\nrightarrow} z \vee z\not\surd)}{x\not\surd}$$

for

$$\frac{\forall_y(\neg\, x \xrightarrow{\tau} y) \wedge x\surd}{x\not\surd} \qquad \frac{\exists_y(x \xrightarrow{\tau} y) \wedge \forall_y(x \xrightarrow{\tau} y \Rightarrow y\not\surd)}{x\not\surd}$$

FOL-SOS - Syntax

Mousavi & Reniers have to write

$$\forall_y \frac{x\downarrow \wedge (x \stackrel{\tau}{\nrightarrow} y \vee y \Downarrow)}{x \Downarrow}$$

for

$$\frac{x\downarrow \wedge \forall_y (x \stackrel{\tau}{\rightarrow} y \Rightarrow y \Downarrow)}{x \Downarrow}$$

FOL-SOS - Semantics

Derivability of an literal l given set of assumptions A.

$$A \vdash l$$

if:

- $l \in A$, or
- there is are rule $\frac{\varphi}{a}$, set S and substitution σ with
  - $l = a\sigma$ and
  - $S \models \varphi\sigma$ and
  - for all $b \in S$, $A \vdash b$

FOL-SOS - Semantics

$\{\neg P\} \vdash Q$

$\Leftarrow$ $\quad rule\ \dfrac{\neg P \wedge \exists_x R(x)}{Q}\ and\ S = \{\neg P, R(c)\}$

$\{\neg P\} \vdash \neg P$ and $\{\neg P\} \vdash R(c)$

$\Leftarrow$ $\quad \neg P \in \{\neg P\}\ and\ rule\ \dfrac{\mathrm{true}}{R(c)}\ and\ S = \emptyset$

true

FOL-SOS - Semantics

We write rewrite $A \vDash L$ if $A \vDash l$ for each $l \in L$.

$\langle C, U \rangle$ is a three-valued stable model if

- $a \in C$ if, and only if, $N \vdash a$ for some N with $C \cup U \vDash N$
- $a \in C \cup U$ if, and only if, $N \vdash a$ for some N with $C \vDash N$

(N is a set of negative literals here.)

FOL-SOS - Semantics

Trivial translation of traditional to FOL-SOS:

$$\frac{S}{a} \quad \mapsto \quad \frac{\bigwedge S'}{a}$$

S' is S with all $t \overset{a}{\nrightarrow}$ replaced by $\forall_x(\neg\, t \overset{a}{\rightarrow} x)$

FOL-SOS - Semantics

From FOL-SOS to traditional SOS is possible for well-defined specifications.

Trivially: $\langle C, \emptyset \rangle$ gives $\{\frac{}{a} \;:\; a \in C\}$.

When not well-defined:

$$\frac{}{a \longrightarrow a} \quad \frac{}{b \longrightarrow b} \quad \frac{\neg b \longrightarrow a}{a \longrightarrow b} \quad \frac{\neg a \longrightarrow b}{b \longrightarrow a}$$

Not possible with just $a \not\twoheadrightarrow$ and $b \not\twoheadrightarrow$.

FOL-SOS - Semantics

Mousavi & Reniers:

For each deduction rule r of the following form,

$$\exists_{\widetilde{z_0}} \forall_{\widetilde{z_1}} \exists_{\widetilde{z_2}} \frac{\bigvee_{i \in I} \bigwedge_{j \in J} \phi_{ij}}{t \xrightarrow{l} t'}$$

$sk(r)$ is $sk(r, \sigma_0, \sigma_{10}, \ldots, \sigma_{20}, \ldots, i_0, \ldots \mid i_j)$ for each substitution $\sigma_0 : \widetilde{z_0} \rightarrow \mathbb{C}$, substitutions $\sigma_{10}, \sigma_{11}, \ldots : \widetilde{z_1} \rightarrow \mathbb{C}$ such that for each $z \in \widetilde{z_1}$, $\{\sigma_{10}(z), \sigma_{11}(z), \ldots\} = \mathbb{C}$, substitutions $\sigma_{20}, \sigma_{21}, \ldots : \widetilde{z_2} \rightarrow \mathbb{C}$, indices $i_0, i_1, \ldots \in I$ and each $i_j \in \{i_0, i_1, \ldots\}$ which is defined as follows.

$$\frac{(\bigwedge_{j \in J} \sigma_0 \cdot \sigma_{10} \cdot \sigma_{20} \phi_{i_0 j}) \wedge (\bigwedge_{j \in J} \sigma_0 \cdot \sigma_{11} \cdot \sigma_{21} \phi_{i_1 j}) \wedge \ldots}{\sigma_0 \cdot \sigma_{1 i_j} \cdot \sigma_{2 i_j} (t \xrightarrow{l} t')}$$

FOL-SOS - Well-Definedness

How to define stratifications w.r.t. formulae?

Looking at semantics: every S such that $S \vDash \varphi$.

Such sets S take role of premises in derivations.

But how to obtain such sets?

FOL-SOS - Well-Definedness

Easier: use set of literals that occur in φ:

$$
\begin{array}{lcl}
\mathrm{Lit}(a) & = & \{a\} \\
\mathrm{Lit}(\neg\varphi) & = & \overline{\mathrm{Lit}}(\varphi) \\
\mathrm{Lit}(\bigwedge\Phi) & = & \bigcup_{\varphi\in\Phi}\mathrm{Lit}(\varphi) \\
\mathrm{Lit}(\forall_x\varphi) & = & \bigcup_t \mathrm{Lit}(\varphi[t/x])
\end{array}
$$

$$
\begin{array}{lcl}
\overline{\mathrm{Lit}}(a) & = & \{\neg a\} \\
\overline{\mathrm{Lit}}(\neg\varphi) & = & \mathrm{Lit}(\varphi) \\
\overline{\mathrm{Lit}}(\bigwedge\Phi) & = & \bigcup_{\varphi\in\Phi}\overline{\mathrm{Lit}}(\varphi) \\
\overline{\mathrm{Lit}}(\forall_x\varphi) & = & \bigcup_t \overline{\mathrm{Lit}}(\varphi[t/x])
\end{array}
$$

FOL-SOS - Well-Definedness

A stratification is a function f of positive atoms to a set S if

- S is well-ordered
  (i.e. there is no infinite sequence $s > s' > \ldots$)
- for all rules $\frac{\varphi}{a}$ and substitutions σ we have that
  - $b \in \mathrm{Lit}(\varphi)$ implies $f(b\sigma) \leq f(a\sigma)$
  - $\neg b \in \mathrm{Lit}(\varphi)$ implies $f(b\sigma) < f(a\sigma)$

If there is a stratification, then the model is two-valued.

FOL-SOS - Well-Definedness

$$\frac{x \downarrow \wedge \forall_y (x \xrightarrow{\tau} y \Rightarrow y \Downarrow)}{x \Downarrow}$$

$$\mathrm{Lit}(x \downarrow \wedge \forall_y (x \xrightarrow{\tau} y \Rightarrow y \Downarrow)) = \{x \downarrow\} \cup \bigcup_t \{\neg x \xrightarrow{\tau} t, t \Downarrow\}$$

$$f(t \downarrow) = 0 \qquad f(t \xrightarrow{\tau} u) = 0 \qquad f(t \Downarrow) = 1$$

FOL-SOS - Congruence Format

$$\frac{\varphi}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

1. The right-hand sides of literals in φ are distinct variables different from $x_1, \ldots, x_n$;
2. the right-hand sides of positive literals in φ are existentially bound;
3. the right-hand sides of negative literals in φ are universally bound;
4. the right-hand side variable of positive literals are bound inside the scope of the variables of the left-hand side of that literal.

FOL-SOS - Congruence Format

$$\frac{\varphi}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

The right-hand sides of literals in φ are distinct variables different from $x_1, \ldots, x_n$.

$$x_1 \xrightarrow{a} y \wedge y \xrightarrow{b} z$$

$$x_1 \xrightarrow{a} x_2$$

$$\exists_{x_3}(x_1 \xrightarrow{a} x_3) \vee \forall_{x_3}(x_2 \xrightarrow{a} x_3)$$

$$\exists_y(x_1 \xrightarrow{a} y \vee \forall_y(x_2 \xrightarrow{a} y))$$

$$\exists_y(x_1 \xrightarrow{a} y \vee x_2 \xrightarrow{a} y)$$

FOL-SOS - Congruence Format

$$\mathrm{dv}(\varphi) \;\wedge\; \mathrm{ubrhs}(\varphi) \cap \{x_1, \ldots, x_n\} = \emptyset$$

$$\begin{array}{lcl}
\mathrm{dv}(t \longrightarrow u) & = & u \in \mathbb{V} \\
\mathrm{dv}(\neg\psi) & = & \mathrm{dv}(\psi) \\
\mathrm{dv}(\bigwedge \Psi) & = & \forall_{\psi \in \Psi} \mathrm{dv}(\psi) \wedge \\
 & & \forall_{\psi,\psi' \in \Psi} (\psi \neq \psi' \;\Rightarrow\; \mathrm{ubrhs}(\psi) \cap \mathrm{ubrhs}(\psi') = \emptyset) \\
\mathrm{dv}(\forall_x \psi) & = & \mathrm{dv}(\psi)
\end{array}$$

$$\begin{array}{lcl}
\mathrm{ubrhs}(t \longrightarrow u) & = & \mathrm{var}(u) \\
\mathrm{ubrhs}(\neg\psi) & = & \mathrm{ubrhs}(\psi) \\
\mathrm{ubrhs}(\bigwedge \Psi) & = & \bigcup_{\psi \in \Psi} \mathrm{ubrhs}(\psi) \\
\mathrm{ubrhs}(\forall_x \psi) & = & \mathrm{ubrhs}(\psi) \setminus \{x\}
\end{array}$$

FOL-SOS - Congruence Format

$$\frac{\varphi}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

The right-hand sides of positive literals in φ are existentially bound.

$x_1 \xrightarrow{a} y$

$\exists_y (x_1 \xrightarrow{a} y)$

$\forall_y (x_1 \xrightarrow{a} y)$

$\forall_y (x_2 \xrightarrow{a} y \Rightarrow x_1 \xrightarrow{a} z)$

$\forall_y (x_2 \xrightarrow{a} z \Rightarrow x_1 \xrightarrow{a} y)$

FOL-SOS - Congruence Format

$$\mathrm{ext}_{\mathrm{FV}(\varphi)\setminus\{x_1,\ldots,x_n\}}(\varphi)$$

$$\begin{aligned}
\mathrm{ext}_S(t \longrightarrow u) &= u \in S \\
\mathrm{ext}_S(\neg\psi) &= \overline{\mathrm{ext}}_S(\psi) \\
\mathrm{ext}_S(\textstyle\bigwedge \Psi) &= \forall_{\psi\in\Psi}\mathrm{ext}_S(\psi) \\
\mathrm{ext}_S(\forall_x\psi) &= \mathrm{ext}_{S\setminus\{x\}}(\psi)
\end{aligned}$$

$$\begin{aligned}
\overline{\mathrm{ext}}_S(t \longrightarrow u) &= \mathrm{true} \\
\overline{\mathrm{ext}}_S(\neg\psi) &= \mathrm{ext}_S(\psi) \\
\overline{\mathrm{ext}}_S(\textstyle\bigwedge \Psi) &= \forall_{\psi\in\Psi}\overline{\mathrm{ext}}_S(\psi) \\
\overline{\mathrm{ext}}_S(\forall_x\psi) &= \overline{\mathrm{ext}}_{S\cup\{x\}}(\psi)
\end{aligned}$$

FOL-SOS - Congruence Format

$$\frac{\varphi}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

The right-hand sides of negative literals in φ are universally bound.

$\neg x_1 \xrightarrow{a} y$

$\forall_y (\neg x_1 \xrightarrow{a} y)$

$\forall_y (x_2 \xrightarrow{a} y \Rightarrow x_1 \xrightarrow{a} z)$

$\forall_y (x_2 \xrightarrow{a} z \Rightarrow x_1 \xrightarrow{a} y)$

FOL-SOS - Congruence Format

$$\text{ext}_{\emptyset}(\neg\varphi)$$

FOL-SOS - Congruence Format

$$\frac{\varphi}{f(x_1, \ldots x_n) \xrightarrow{a} t},$$

The right-hand side variable of positive literals are bound inside the scope of the variables of the left-hand side of that literal.

$\exists_z(\forall_y(y \xrightarrow{a} z))$

$\forall_y(\exists_z(y \xrightarrow{a} z))$

$\exists_y(\exists_z(z \xrightarrow{a} y))$

$\exists_y(\forall_w(\exists_z(z \xrightarrow{a} y)))$

FOL-SOS - Congruence Format

$$h^{\mathrm{FV}(\varphi)\setminus\{x_1,\ldots,x_n\}}(\varphi)$$

$$\begin{array}{rcl} h^S(t \to u) & = & \mathrm{true} \\ h^S(\neg\varphi) & = & \overline{h}^S(\varphi) \\ h^S(\bigwedge \Phi) & = & \forall_{\varphi\in\Phi} h^S(\varphi) \\ h^S(\forall_x \varphi) & = & h^{\emptyset}(\varphi) \ \wedge \ k^S_{\{x\}}(\varphi) \end{array} \qquad \begin{array}{rcl} \overline{h}^S(t \to u) & = & \mathrm{true} \\ \overline{h}^S(\neg\varphi) & = & h^S(\varphi) \\ \overline{h}^S(\bigwedge \Phi) & = & \forall_{\varphi\in\Phi} \overline{h}^S(\varphi) \\ \overline{h}^S(\forall_x \varphi) & = & \overline{h}^{S\cup\{x\}}(\varphi) \end{array}$$

$$\begin{array}{rcl} k^S_T(t \to u) & = & (u \in S \setminus T) \Rightarrow (\mathrm{var}(t) \cap T = \emptyset) \\ k^S_T(\neg\varphi) & = & k^S_T(\varphi) \\ k^S_T(\bigwedge \Phi) & = & \forall_{\varphi\in\Phi} k^S_T(\varphi) \\ k^S_T(\forall_x \varphi) & = & k^S_{T\cup\{x\}}(\varphi) \end{array}$$

FOL-SOS - Congruence Format

The examples all fit our congruence format.

$$\frac{\bigwedge_{a \in A}(\forall_y(\neg\, x \xrightarrow{a} y) \wedge \neg\, x \xrightarrow{a} \surd)}{\delta(x)}$$

$$\frac{\forall_y(\neg\, x \xrightarrow{\tau} y) \wedge x\surd}{x \surd\!\!\!\surd} \qquad \frac{\exists_y(x \xrightarrow{\tau} y) \wedge \forall_y(x \xrightarrow{\tau} y \Rightarrow y \surd\!\!\!\surd)}{x \surd\!\!\!\surd}$$

$$\frac{x \downarrow \wedge \forall_y(x \xrightarrow{\tau} y \Rightarrow y \Downarrow)}{x \Downarrow}$$

FOL-SOS - Congruence Format

Traditional format is incorporated:

1. The right-hand sides of literals in φ are distinct variables different from $x_1, \ldots, x_n$; $\Leftarrow$ traditional distinctness requirement
2. the right-hand sides of positive literals in φ are existentially bound; $\Leftarrow$ trivially
3. the right-hand sides of negative literals in φ are universally bound; $\Leftarrow$ only quantifications: $\forall_x(\neg\, t \xrightarrow{a} x)$
4. the right-hand side variable of positive literals are bound inside the scope of the variables of the left-hand side of that literal. $\Leftarrow$ no (nested) quantifications

FOL-SOS - Congruence Format

Mousavi & Reniers format (UNTyft/UnTyxt) is incorporated:

1. The right-hand sides of literals in φ are distinct variables different from $x_1, \ldots, x_n$; $\Leftarrow$ same
2. the right-hand sides of positive literals in φ are existentially bound; $\Leftarrow$ same
3. the right-hand sides of negative literals in φ are universally bound; $\Leftarrow$ same
4. the right-hand side variable of positive literals are bound inside the scope of the variables of the left-hand side of that literal. $\Leftarrow$ simpler variant due to $\exists_{\vec{x}} \forall_{\vec{y}} \exists_{\vec{z}}$

Outline

Summary

- Full first-order power in premises of rules
- Straightforward extension of semantics
- Conservative extension of traditional SOS (with sugar)
- Conservative extension of traditional congruence format
- Congruence format requirements easily calculable
- Suits known examples using quantifications

Thank you for your attention!